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MULTIPLE-PRODUCTION THEORY VIA TOLLER VARIABLES*

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Toller's group-theoretical analysis of kinematics is exploited to define a complete set of variables, each of independent range, for particle production of arbitrary multiplicity. In terms of these variables, the generalized Regge-pole hypothesis leads to a simple, unambiguous, and experimentally accessible prediction for high-energy multiple-production cross section. A flat Pomeranchuk trajectory is shown to violate the Froissart bound.

A variety of multiperipheral models for inelastic reactions at high energy has been discussed in the literature,[1-7] but the implementing variables have been incomplete or imperfectly matched to the factorizability which characterizes such models. In this paper we exploit the work of Toller[8] to define a complete set of variables for particle production of arbitrary multiplicity, the range of each variable being independent of the others. The new variable set is natural for the implementation of any multiperipheral model, leading to a phase space that factors asymptotically in the same manner as does the amplitude. We apply our variables to the (unique) generalization of the Regge-pole hypothesis, achieving a simple, unambiguous, and experimentally accessible prediction for multiple-production cross sections at high energy which maintains the factorization property. One important aspect of the result is the exclusion of the possibility of a flat Pomeranchuk trajectory.

For the N-particle production reaction $a+b \rightarrow 1+2+\cdots+N$, we begin by selecting a particular ordering of final particles so as to define a set of $N-1$ momentum transfers Q_{mn} according to the diagram of Fig. 1. Each different ordering leads to a different set of variables; any of these sets is complete, the choice between them being a matter of convenience usu-

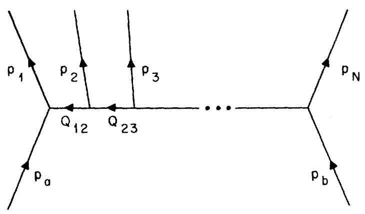

FIG. 1. Diagram defining the momentum transfers Q_{nm}.

ally resolved by appeal to the multiperipheral concept. That is to say, for describing a particular region of final-particle momenta, one generally chooses that set of variables for which all Q_{mn}^2 in this region are small, while all $s_{mn}=(p_m+p_n)^2$ are large.

The number of variables needed to describe an amplitude with a total of $N+2$ ingoing and outgoing particles is well known to be $3N-4$, once Lorentz invariance is included. We divide the total variable set into three categories, a set of $N-1$ t variables, a set of $N-1$ ξ variables, and a set of $N-2$ ω variables. This choice is motivated in detail by Bali, Chew, and Pignotti[9] on the basis of Toller's group-theoretical analysis.[8] The t variables are obvious: $t_{mn}=Q_{mn}^2$. Less obvious but still recognizable are the ξ_{mn}, which are also in one-to-one correspondence with the Q_{mn}, $i\xi_{mn}$ being the analytic continuation of the angle in the rest system of Q_{mn} between the direction of $\vec{p}_m$ and that of $\vec{p}_n$. In the region of interest here the Q_{mn} are spacelike (the t_{mn} are negative), and Toller has shown that each ξ_{mn} is real, ranging from 0 to ∞ independently of the other variables.

The ω_n are the least familiar components of our variable set. The members of this subset are in one-to-one correspondence with the internal vertices of Fig. 1. To understand ω_n, go into the rest frame of p_n, where the spatial components of the two adjacent momentum transfers point in the same direction. Then consider the rigid rotation about this axis of all momenta standing on the left of the vertex n and the independent rigid rotation of all momenta standing on the right. The difference of these two-rotation angles is ω_n, which thus has a range 0 to 2π.

In Ref. 9 it is shown explicitly how to pass by a succession of Lorentz transformations from the variables t_{mn}, ξ_{mn}, ω_n to the ordinary momentum variables or to the channel invariants $s_{mn}=(p_m+p_n)^2$ and $s=(p_a+p_b)^2$. In the interest of brevity we confine ourselves here to the observation that s_{mn} is a linear function of $\cosh\xi_{mn}$, with coefficients that depend only on the t's adjacent to the m and n vertices:

$$s_{mn}=[\lambda_m^{1/2}\lambda_n^{1/2}/2(-t_{mn})]\cosh\xi_{mn}+\text{function of } t\text{'s}, \tag{1}$$

with $\lambda_n=\lambda(m_n^2, t_{n-1,n}, t_{n,n+1})$ (for the end vertices, one of the t's should be replaced by m_a^2 or m_b^2), where

$$\lambda(t_i,t_j,t_k)=t_i^2+t_j^2+t_k^2-2t_it_j-2t_it_k-2t_jt_k. \tag{2}$$

Thus a large value of ξ_{mn}, with adjacent t's small, implies a large value of s. It turns out that s depends on all $3N-4$ variables, but when all the $\cosh\xi_{mn}$ are large,

$$s\sim\frac{\lambda_1^{1/2}\lambda_N^{1/2}}{2(-t_{12})^{1/2}(-t_{N-1,N})^{1/2}}\cosh\xi_{12}\cosh\xi_{23}\cdots\cosh\xi_{N-1,N}\prod_{i=2}^{N-1}(\cos\omega_i+\cos q_i), \tag{3}$$

where

$$\cosh q_n=\frac{m_n^2-t_{n-1,n}-t_{n,n+1}}{2(-t_{n-1,n})^{1/2}(-t_{n,n+1})^{1/2}}. \tag{4}$$

The $3N-4$ dimensional phase spac

$$d\Phi_N=d^4p_1\delta(p_1^2-m_1^2)d^4p_2^2\delta(p_2^2-m_2^2)\cdots d^4p_N\delta(p_N^2-m_N^2)\delta^4(p_1+p_2+\cdots+p_N-p_a-p_b),$$

in terms of the new variables becomes

$$d\Phi_N = \frac{2}{m_a^2 m_b^2} \frac{\lambda_1^{1/2}\lambda_2^{1/2}\cdots\lambda_N^{1/2}}{2^{3N}(-t_{12})(-t_{23})\cdots(-t_{N-1,N})} dt_{12}dt_{23}\cdots dt_{N-1,N}$$
$$\times d\cosh\xi_{12} d\cosh\xi_{23}\cdots d\cosh\xi_{N-1,N} d\omega_2 d\omega_3 \cdots d\omega_{N-1} d\psi \frac{\delta(\cosh\eta - p_a\cdot p_b/m_a m_b)}{\sinh\eta}, \quad (5)$$

the angle ψ describing rigid rotations of the entire final set of N momenta about the common direction of p_a and p_b in a frame where these initial momenta are parallel. Evidently the spin-averaged matrix element will not depend on ψ, but it will in general depend on all the other variables appearing in formula (5). For a target at rest, $\cosh\eta$ is the energy of the incident particle in units of its own rest mass. In terms of s,

$$\cosh\eta = \frac{s - m_a^2 - m_b^2}{2m_a m_b} \sim \frac{s}{2m_a m_b}.$$

The single constraint interlocking our variables arises through the delta function in $\cosh\eta$. It follows, however, from formula (3) that when all the $\cosh\eta_{mn}$ are large,

$$\cosh\eta \sim (\text{factorizable function of } t_{mn} \text{ and } \omega_n)\cosh(\xi_{12}+\xi_{23}+\cdots\xi_{N-1,N}), \quad (3')$$

so that for a fixed set of t's and ω's the constraint is only on the sum of the ξ's. It is typical of multiperipheral models that when η is large, most of the production occurs in regions where every ξ_{mn} is large. Thus the approximation (3′) can be used to simplify the phase space:

$$d\Phi_N \approx \frac{\text{factorizable function of } t\text{'s and } \omega\text{'s}}{\sinh\eta} \delta(\xi_{12}+\xi_{23}+\cdots+\xi_{N-1,N}-\xi^{(+)})$$
$$\times dt_{12}\cdots dt_{N-1,N} d\xi_{12} d\xi_{23}\cdots d\xi_{N-1,N} d\omega_2 d\omega_3 \cdots d\omega_{N-1} d\psi, \quad (6)$$

where, from formula (3),

$$\cosh\xi^{(+)} \approx s \frac{2(-t_{12})^{1/2}(-t_{N-1,N})^{1/2}}{\lambda_1^{1/2}\lambda_N^{1/2}} 2^{N-2} \prod_{i=2}^{N-1} (\cos\omega_i + \cos q_i)^{-1}. \quad (7)$$

We are now in a position to write down a cross-section formula. Suppose, for example, that the Regge-pole hypothesis is adopted for the absolute square of the amplitude, summed over final spins and averaged over initial spins[3-7,9]:

$$\langle |A(t_{mn},\xi_{mn},\omega_n)|^2\rangle_{\text{Av}} \sim f_1(t_{12}) f_2(t_{12},\omega_2,t_{23})\cdots f_N(t_{N-1,N})(\cosh\xi_{12})^{2\alpha_{12}(t_{12})}(\cosh\xi_{23})^{2\alpha_{23}(t_{23})}\cdots. \quad (8)$$

The internal "vertex functions" f_n describe the coupling of two Regge trajectories to a physical particle, while f_1 and f_N couple two physical particles to a single Regge trajectory. Taken together with formula (6) and the flux factor, and integrating over $d\psi$, the behavior (8) leads to

$$s^2 d\sigma_N \sim F_1(t_{12})F_N(t_{N-1,N}) \prod_{i=2}^{N-2} F_i(t_{i-1,i},\omega_i,t_{i,i+1}) \exp[2\alpha_{12}(t_{12})\xi_{12}]$$
$$\times \exp[2\alpha_{23}(t_{23})\xi_{23}]\cdots dt_{12}dt_{23}\cdots d\xi_{12}d\xi_{23}\cdots d\omega_2\cdots d\omega_{N-1}\delta[\xi_{12}+\xi_{23}+\cdots-\xi^{(+)}], \quad (9)$$

a result containing a wealth of physically interesting predictions, especially if one exploits the correlations between different reactions flowing from the universality of the vertex functions F_n. (The function f_n differs from F_n only by a factor that depends in a known manner on $t_{n-1,n}$, $t_{n,n+1}$, and ω_n.)

We make no attempt here to exhaust the content of formula (9), but three of the most obvious features are:

(a) Consider a reaction in which all leading trajectories are the Pomeranchuk and suppose this trajectory to be perfectly flat (a fixed pole) at $\alpha = 1$. Then

$$s^2 d\sigma_N \sim \prod_{i=1}^{N} F_i e^{2\xi^{(+)}} dt_{12} dt_{23} \cdots d\xi_{12} d\xi_{23} \cdots d\omega_2 \cdots d\omega_{N-1} \delta[\xi_{12} + \xi_{23} \cdots - \xi^{(+)}],$$

or, integrating over the $d\xi$'s and remembering (7),

$$d\sigma_N \sim (\text{function of } t\text{'s and } \omega\text{'s})(\ln s)^{N-2} dt_{12} dt_{23} \cdots dt_{N-1,N} d\omega_2 \cdots d\omega_{N-1}. \tag{10}$$

The limits on the t and ω intervals become independent of s for large s; so there is a conflict with the Froissart limit[10] for $N > 4$, showing that peak shrinkage, such as that associated with a moving pole, is essential to the consistency of the model.

(b) Assuming all poles to move, if formula (9) is integrated over the $d\xi$'s, we find that

$$s^2 d\sigma_N \sim \frac{\Pi_i F_i}{2^{N-2}} \left\{ \frac{\exp[2\alpha_{12}\xi^{(+)}]}{(\alpha_{12}-\alpha_{23})(\alpha_{12}-\alpha_{34})\cdots(\alpha_{12}-\alpha_{N-1,N})} + \frac{\exp[2\alpha_{23}\xi^{(+)}]}{(\alpha_{23}-\alpha_{12})(\alpha_{23}-\alpha_{34})\cdots} + \cdots \frac{\exp[2\alpha_{N-1,N}\xi^{(+)}]}{(\alpha_{N-1,N}-\alpha_{12})\cdots} \right\} dt_{12} dt_{23} \cdots d\omega_2 d\omega_3 \cdots. \tag{11}$$

The energy dependence of this differential cross section is

$$\propto s^{2[\alpha_{\max}-1]},$$

where $\alpha_{\max}$ is the highest trajectory in the chain. Such a dependence was conjectured by Zachariasen and Zweig.[7]

(c) For processes in which one trajectory in the chain lies well below the others, the differential final-particle spectrum will favor low subenergies for the particle pair corresponding to the low-lying trajectory. That is to say, even at a fixed incident energy it is possible to investigate the characteristic Regge structure by studying ratios of final subenergies. The logarithmic distribution in the ratio of two subenergies, keeping all other ratios fixed, is predicted by formula (9) to be a straight line whose slope is determined by the difference of the corresponding trajectory heights. In particular, for $N = 3$ after integrating over $d\omega_2$,

$$d\sigma_3 \sim (\text{function of } t\text{'s})\, s^{\alpha_{12}(t_{12}) + \alpha_{23}(t_{23}) - 2} (s_{12}/s_{23})^{\alpha_{12}(t_{12}) - \alpha_{23}(t_{23})} dt_{12} dt_{23} d\ln(s_{12}/s_{23}). \tag{13}$$

A concluding remark is that clusters of final particles with low total mass can replace any or all of the single outgoing particles in Fig. 1, so long as the experimenter sums over degrees of freedom within a given cluster—except for the cluster mass m_n. The general problem is analyzed in detail in Ref. 9. It turns out that all formulas given in this paper continue to hold, with single-particle masses replaced by cluster masses.

Questions involving total cross sections and the over-all multiplicity of production require integration over the dt's and $d\omega$'s as well as a summation over final-particle combinations. These matters will be considered in another paper.

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RELATIVISTICALLY INVARIANT SOLUTIONS OF CURRENT ALGEBRAS AT INFINITE MOMENTUM*

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Recently, following a suggestion by Fubini and Furlan,[1] infinite-momentum limits of weak and electromagnetic currents between one-particle states have attracted much attention. The mathematical properties of these limits have been investigated by Coester and Roepstorff.[2] According to these authors, the assumption that one-particle matrix elements of current algebras are saturated by one-particle intermediate states at infinite momentum, a suggestion particularly advocated by Dashen and Gell-Mann,[3] is incompatible with Lorentz invariance unless an infinite sequence of resonances with arbitrarily high spin values is involved. We wish to show that if such an infinite sequence of Regge-like recurrences is taken into account, there do indeed exist nontrivial solutions which are compatible with Lorentz invariance.

Matrix elements at infinite momentum.—We start with the following simple remark. Let $F^\mu(x)$ be a vector or axial-vector current and consider the matrix element

$$L^\mu = \lim_{\kappa\to\infty} \langle p_\kappa', N' | F^\mu(0) | p_\kappa N\rangle, \qquad (1)$$

where p_κ' and p_κ denote the four-momenta of the one-particle states of mass m' and m, respectively,

$$p_\kappa' = (\omega_\kappa', \vec{p}\,' + \kappa\vec{a}); \quad p_\kappa = (\omega_\kappa, \vec{p} + \kappa\vec{a}).$$

Here $\vec{a}$ is a unit vector pointing in the z direction. The symbols N' and N denote the remaining quantum numbers like spin and charge. The states $|p, N\rangle$ are obtained from states at rest by means of a pure Lorentz transformation

$$|p, N\rangle = U[L(p)]\,|m, N\rangle (m/\omega)^{1/2},$$

where $L(p)$ is the transformation that takes the vector $m^\mu = (m, 0, 0, 0)$ to the vector p^μ. This allows one to express the limit L^μ as a matrix element between states at rest:

$$L^\mu = \lim_{\kappa\to\infty} \left(\frac{mm'}{\omega_\kappa \omega_\kappa'}\right)^{1/2} \langle m', N' | U[L^{-1}(p_\kappa')L(p_\kappa)] L_\nu{}^\mu(p_\kappa) F^\nu(0) | mN\rangle.$$

The product $L^{-1}(p_\kappa')L(p_\kappa)$ has a finite limit, and furthermore,

$$\lim_{\kappa\to\infty} L_\nu{}^\mu(p_\kappa) \left(\frac{mm'}{\omega_\kappa\omega_\kappa'}\right)^{1/2} = \left(\frac{m'}{m}\right)^{1/2} a^\mu \bar{a}_\nu, \qquad (2)$$

where a^μ and $\bar{a}^\mu$ denote the two lightlike vectors $a^\mu = (1, \vec{a})$ and $\bar{a}^\mu = (1, -\vec{a})$. The limit of the product $L^{-1}(p_\kappa')L(p_\kappa)$ may be expressed in terms of the 2×2 representation of the homogeneous Lorentz group as follows:

$$K = \lim_{\kappa\to\infty} L^{-1}(p_\kappa')L(p_\kappa) = (m'm)^{-1/2} \begin{pmatrix} m' & 0 \\ -q & m \end{pmatrix}.$$

The complex number q stands for $q = q^1 + iq^2$, where q^1 and q^2 are the components of the momentum transfer $\vec{q} = \vec{p}\,' - \vec{p}$ in the x-y plane. We represent the matrix K as a product

$$K = L^{-1}(\pi')QL(\pi), \qquad (3)$$

where the matrices $L(\pi')$, $L(\pi)$ represent pure Lorentz transformations in the z direction that take the vectors m', m to π', π, respectively,

$$\pi^\mu = (2m_0)^{-1}[m_0^2 + m^2, (m_0^2 - m^2)\vec{a}];$$
$$\pi^{\mu\prime} = (2m_0)^{-1}[m_0^2 + m'^2, (m_0^2 - m'^2)\vec{a}]. \qquad (4)$$